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LETTER TO THE EDITOR

Solutions to Frenkel's deformation of the KP hierarchy

Plamen Iliev†

Institut de Mathématique, Université Catholique de Louvain, Chemin du Cyclotron, 2 1348 Louvain-la-Neuve, Belgique

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Abstract. We show that any solution of the KP hierarchy leads to a solution of Frenkel's hierarchy. This allows us to make an explicit relation between his hierarchy and the one introduced by Haine and Iliev in 1997.

The first q-deformations of the KdV hierarchy and related soliton equations were introduced by Frenkel [4]. He showed that these hierarchies are bi-Hamiltonian and connected them with q-deformations of the W-algebras investigated in [5].

In [6] Haine and Iliev considered a slightly different deformation of the KP hierarchy and showed that, by making an appropriate shift in the arguments of the classical Schur polynomials, one obtains rational solutions of the deformed hierarchy. This result was independently extended in [1, 7] where it was proved that the same shift in any tau-function gives a solution to the deformed hierarchy. The proof in [1] was based on the connection between the solutions of the KP hierarchy and 1-Toda lattice equations, while in [7] we used a q-version of Sato theory [8, 2, 9, 3].

The aim of this letter is to show that a simple translation with respect to the $t_i's$ connects the tau-functions of these two models. Tau-functions solutions to Frenkel's deformation are constructed as deformations of the classical tau-functions. For details we refer the reader to [7], where we used the same scheme, based on the approach of Dickey [3].

We shall denote by D and Δ_q, respectively, the q-shift and q-difference operators, acting on functions of x by

$$Df(x) = f(xq) \tag{1}$$

$$\Delta_q f(x) = f(xq) - f(x). \tag{2}$$

Using the fact that for $n \in \mathbb{Z}_+$

$$\Delta_q^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^{n-k}\Delta_q^k f)\Delta_q^{n-k} g$$

one can define $\Delta_q^n \cdot f$ for any $n \in \mathbb{Z}$ as the formal operator

$$\Delta_q^n \cdot f = \sum_{k=0}^{\infty} \binom{n}{k} (D^{n-k}\Delta_q^k f)\Delta_q^{n-k}. \tag{3}$$

† E-mail address: iliev@agel.ucl.ac.be

As in the classical case it is not difficult to see that the formal q-difference operators form an associative algebra.

We define the formal adjoint to the operator $P = \sum a_i \Delta_q^i$ to be $P^* = \sum \Delta_{1/q}^i \cdot a_i$. One can easily deduce the usual identity

$$(PQ)^* = Q^* P^* \tag{4}$$

for any pseudo-difference operators P and Q.

Now, we shall briefly recall the Frenkel's deformation of the Kadomtsev–Petviashvili hierarchy (q-KP). We refer the reader to [4] for details as well as for a description of the bi-Hamiltonian structure of the deformed hierarchy. Consider the formal q-pseudo-difference operator

$$L = \Delta_q + a_0 + \sum_{i=1}^{\infty} a_i \Delta_q^{-i}. \tag{5}$$

The q-deformed KP hierarchy is defined by the Lax equations

$$\frac{\partial L}{\partial t_j} = [L_+^j, L] \tag{6}$$

where as usual L_+^j stands for the pure q-difference part of the operator L^j. Let us denote by $S = 1 + \sum_{k=1}^{\infty} w_k \Delta_q^{-k}$ the wave operator, i.e. the operator which conjugates L to Δ_q

$$L = S \Delta_q S^{-1}. \tag{7}$$

The vector fields can be extended by

$$\frac{\partial S}{\partial t_j} = -L_-^j S. \tag{8}$$

We shall need the following 'q-exponential' function

$$E_q(x, z) = \exp\left(\frac{\log x \log(1+z)}{\log q}\right). \tag{9}$$

Clearly,

$$\Delta_q E_q = z E_q \qquad \text{and} \qquad \Delta_q^* E_{1/q} = z E_{1/q}.$$

For a q-pseudo-difference operator $P = \sum p_i \Delta_q^i$ we introduce the following notation

$$P|_{x/t} = \sum p_i(x/t) \Delta_q^i$$

which corresponds to the linear change of variable $y = x/t$ in the operator P. If we denote

$$\mathrm{res}_z\left(\sum a_i z^i\right) = a_{-1} \qquad \text{and} \qquad \mathrm{res}_{\Delta_q}\left(\sum b_j \Delta_q^j\right) = b_{-1}$$

one can check that for any two operators P and Q we have the following q-analogue of a lemma due to Dickey [3]:

$$\mathrm{res}_z(P E_q(x, z) Q^*|_{x/q} E_{1/q}(x, z)) = \mathrm{res}_{\Delta_q}(PQ). \tag{10}$$

Let us define the q-wave function w_q and the q-ajoint wave function w_q^* for the q-KP hierarchy by

$$w_q = S E_q(x, z) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)$$

and

$$w_q^* = (S^*)^{-1}|_{x/q} E_{1/q}(x, z) \exp\left(-\sum_{i=1}^{\infty} t_i z^i \right).$$

Now we can formulate the q-analogue of the 'bilinear identity' which characterizes the solutions of the deformed hierarchy.

Proposition 1. If L is a solution of the q-KP hierarchy (6), then for any $n \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ multi-index with $\alpha_i \geqslant 0$ we have

$$\mathrm{res}_z(\Delta_q^n \partial^\alpha w_q \ w_q^*) = 0. \tag{11}$$

Conversely, if

$$w_q = \left(1 + \sum_{i=1}^{\infty} w_i z^{-i}\right) E_q(x, z) \exp\left(\sum_{i=1}^{\infty} t_i z^i \right)$$

and

$$w_q^* = \left(1 + \sum_{i=1}^{\infty} w_i^* z^{-i}\right) E_{1/q}(x, z) \exp\left(-\sum_{i=1}^{\infty} t_i z^i \right)$$

are formal series, with w_i and w_i^* functions of $(x, t_1, t_2, \ldots)$ and if (11) holds for any $n \in \mathbb{Z}_+$ and for any multi-index α with nonnegative components α_i, then the operator $L = S\Delta_q S^{-1}$, where $S = 1 + \sum_{i=1}^{\infty} w_i \Delta_q^{-i}$ is a solution to the q-KP hierarchy.

Using this proposition one can express the wave and adjoint wave functions in terms of a q-analogue of the tau-function $\tau_q(x; t)$

$$w_q = \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x; t)} E_q(x; z) \exp\left(\sum_{i=1}^{\infty} t_i z^i \right)$$

$$w_q^* = \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x; t)} E_{1/q}(x, z) \exp\left(-\sum_{i=1}^{\infty} t_i z^i \right)$$

where as usual

$$[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots \right).$$

From the 'bilinear identity' and expanding (9) with respect to z one sees immediately that any solution of the classical KP hierarchy leads to a solution of the deformed hierarchy.

Theorem 2. Let $\tau(t)$ be a tau-function of the classical KP hierarchy. Then

$$\tau_q(x; t) = \tau(t + [x]_q) \tag{12}$$

where

$$[x]_q = \left(\frac{\log x}{\log q}, -\frac{1}{2}\frac{\log x}{\log q}, \frac{1}{3}\frac{\log x}{\log q}, \ldots \right)$$

is a tau-function of the q-KP hierarchy.

Remark. When $\tau(t)$ is a tau-function for a solution of the of the Nth Gel'fand–Dickey hierarchy, then L^N is a q-difference operator of the form

$$L^N = D^N - u_1 D^{N-1} + u_2 D^{N-2} + \cdots + (-1)^N \tag{13}$$

which solves the Nth deformed hierarchy [4]. For example, the second-order operator

$$L = D^2 - (2-u)D + 1 \tag{14}$$

is a solution of the q-KdV hierarchy, where

$$u = \Delta_q \frac{\partial}{\partial t_1} \log \tau_q(x;t)\tau_q(xq;t) \tag{15}$$

and $\tau(t)$ is a tau-function for the classical KdV equation.

The last theorem shows the connection between Frenkel's hierarchy and the one considered in [6]. The formulae

$$t_k' = t_k + \frac{(-1)^k}{k}\frac{\log x}{\log q} + \frac{(1-q)^k}{k(1-q^k)}x^k \tag{16}$$

for $k \in \mathbb{Z}_+$ give the correspondence between the tau-functions of these two models (see [1, 7]). In fact (16) is the transformation which connects the 'exponential functions'.

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